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TEMPERATURE PROFILES FOR TURBULENT FLOW OF HIGH PRANDTL NUMBER FLUIDS†

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Abstract—A comprehensive turbulent transport model has been developed by Harriott which has been found to correlate experimental data for the mean Nusselt number for fluids with a wide range of Prandtl numbers. This surface rejuvenation model is now adapted to the study of temperature or concentration profiles for turbulent pipe flow, with emphasis on high Prandtl (Schmidt) number fluids.

NOMENCLATURE

c_p, specific heat at constant pressure;
d, diameter of tube;
erfc, complement of error function;
f, Fanning friction factor;
h, heat transfer coefficient;
H, eddy approach distance;
K, thermal conductivity;
q_0, heat flux at wall;
t, instantaneous temperature of eddy;
$\bar{t}$, mean temperature of eddies;
T, mean temperature of fluid flow;
u, velocity of eddy;
U^*, friction velocity, $\sqrt{(\sigma_0/\rho)}$;
U, bulk stream velocity;
y, radial distance from the wall.

Greek symbols
α, molecular thermal diffusivity;
ε_h, thermal eddy diffusivity;
ε_m, momentum eddy diffusivity;
θ, instantaneous contact time;
ν, kinematic viscosity;
ρ, density of fluid;
τ, mean residence time;
σ_0, shear stress at wall.

Non-dimensional groupings
H^+, dimensionless approach distance of eddies HU^*/ν;
Nu, Nusselt number, hd/K;
Pr, Prandtl number, $\mu c_p/K$;
Re, Reynolds number, dU/ν;
T^+, non-dimensional temperature parameter, $(T_0 - T)\rho c_p U^*/q_0$;
y^+, non-dimensional distance in radial direction from wall, yU^*/ν.

Subscripts
b, condition at bulk stream;
i, at the first instant of rejuvenation or renewal;
0, condition at wall.

INTRODUCTION

A FAIRLY new class of turbulent transport models are based on the surface renewal principle first set forth by Danckwerts [3]. These surface renewal and penetration models are based on the notion that eddies intermittently move from the turbulent core to the close vicinity of the surface. During the residency of elements of

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fluid within the wall region, simple one-dimensional unsteady molecular transport is assumed to be predominate. (The effect on the mean transport of numerous eddies moving to the surface has generally been accounted for by the use of the age distribution principle.) The most elementary form of the surface renewal and penetration model is based on the assumption that eddies move into direct contact with the surface. This model gives rise to expressions for the mean temperature profile and Nusselt number for a fluid flowing turbulently inside a smooth pipe of the forms [15]

$$\frac{T - T_0}{T_i - T_0} = 1 - \exp\left[-y^+ \sqrt{Pr}\sqrt{(f/2)}\right] \quad (1)$$

or

$$T^+ = \sqrt{Pr}\sqrt{(2/f)}\left\{1 - \exp\left[-y^+ \sqrt{Pr}\sqrt{(f/2)}\right]\right\} \quad (2)$$

and

$$Nu = \frac{T_0 - T_i}{T_0 - T_b}\frac{f}{2} Re \sqrt{Pr} \quad (3)$$

where $T^+ = (T_0 - T)\rho U^* c_p/q_0$ and T_i represents the eddy temperature at the first instant of renewal; T_i may be set equal to the bulk stream temperature. T_b, for fluids other than liquid metals [15].

Experimental data by Fage and Townend [15], Runstadler *et al.* [14], Nedderman [12] and Popovich and Hummel [13] support the basic surface renewal principle. However, the recent investigation by Popovich and Hummel indicates that turbulent eddies do not generally move into direct contact with the surface; these workers reported a dimensionless mean approach distance, H^+, equal to 5·0. Accordingly, the basic surface renewal and penetration model may be expected to become inappropriate for fluids in which the major thermal resistance is concentrated near the transport surface. Indeed, equation (3) (with T_i set equal to T_b) adequately correlates experimental data for fluids with moderate values of the Prandtl number ($0{\cdot}5 < Pr < 5{\cdot}0$), but becomes inadequate for fluids with Prandtl number values very much greater than unity [16]. Likewise equations (1) and (2) become inappropriate for fluids with large values of the Prandtl number. Parenthetically, for fluids with low values of the Prandtl number, T_i may not be set equal to T_b. In this regard, equation (2), which does not contain the parameter T_i, has been found to correlate experimental data for liquid metals, as well as moderate Prandtl number fluids.

Harriott [8] has developed a surface renewal and penetration model for turbulent mass transfer which accounts for the effect of eddies not moving into direct contact with the surface. This surface rejuvenation model is based on the hypothesis that eddies move to within random distances of the surface, remaining for various lengths of time. Calculations based on this model have been shown to be consistent with experimental data for the Nusselt and Sherwood numbers for fluids with large as well as moderate values of the Prandtl and Schmidt numbers [18]. The purpose of this paper is to employ the surface rejuvenation model in the estimation of temperature or concentration profiles associated with turbulent tube flow in order to further demonstrate its usefulness.

MATHEMATICAL FORMULATION OF THE MODEL

Based on the surface rejuvenation model, the following system of equations may be written for any given eddy-rejuvenation cycle:

$$\frac{\partial t}{\partial \theta} = \alpha \frac{\partial^2 t}{\partial y^2} \quad 0 \leqslant y < \infty \quad (4)$$

$$t = T_0 \text{ at } y = 0 \quad (5)$$

for uniform wall temperature, and

$$t = T_i \text{ as } y \to \infty \quad (6)$$

$$t = F(y) \text{ at } \theta = 0 \quad (7)$$

where $F(y)$ represents the temperature profile within the wall region at the first instant of rejuvenation; for an instantaneous approach distance of H,

$$F(y) = T_i \text{ for } y > H$$

such that $F(y)$ is a truncated curve. The solution to these equations may be written as (2)

$$\left[\frac{t - T_i}{T_0 - T_i} = \operatorname{erfc}\frac{y}{2\sqrt{(\alpha\theta)}}\right] + \left[\frac{y}{2\sqrt{(\pi\alpha\theta)}}\right]\int_0^\infty [F(\xi) - T_i]\left[\exp\frac{(y-\xi)^2}{-4\alpha\theta} - \exp\frac{(y+\xi)^2}{-4\alpha\theta}\right] d\xi \tag{8}$$

where the instantaneous time θ is zero at the first instant of rejuvenation and $F(\xi)$ is dependent on the residence time and approach distance for the preceding eddy-rejuvenation cycle.

Based on equation (8), the temperature history associated with an individual rejuvenation cycle may be expressed as

$$\frac{t(y, \tau_k) - T_i}{T_0 - T_i} = \operatorname{erfc}\left[\frac{y}{2\sqrt{(\alpha\tau_k)}}\right] + \frac{y}{2\sqrt{(\pi\alpha\tau_k)}}\int_0^\infty [F(\xi) - T_i]\left[\exp\frac{(y-\xi)^2}{-4\alpha\tau_k} - \exp\frac{(y+\xi)^2}{-4\alpha\tau_k}\right] d\xi \tag{9}$$

where $\tau_k = k\Delta\tau$ and k assumes integer values ranging from zero to $\tau/\Delta\tau$. An expression for the time average temperature profile associated with each eddy-rejuvenation cycle, $\bar{t}$, can now be written as

$$\bar{t} = \frac{1}{\tau}\sum_{k=0}^{\tau/\Delta\tau} t(y, \tau_k)\,\Delta\tau. \tag{10}$$

In a calculation procedure which is synonymous to that used by Harriott for the mean transfer flux, calculations may be made for the mean temperature profile, T, by the generation of a sequence of approach distances H_1, H_2, H_3, etc., on a digital computer. Assuming that the first rejuvenation cycle involves an eddy which moves into direct contact with the wall, the calculations for t and $\bar{t}$ were started by setting $F(y)$ equal to T_i for $y > 0$. The starting condition for the calculations associated with the second eddy-rejuvenation cycle is

$$t = F(y, \tau) \text{ for } y < H_2$$
$$t = T_i \quad \text{ for } y > H_2$$

where the function F represents the calculations for t at the end of the preceding rejuvenation cycle ($\tau_k = \tau$). The calculations proceed accordingly with the time average temperature profile calculated for each eddy-rejuvenation cycle. After each cycle, the mean temperature profile for all previous intervals is calculated; the calculations are continued until the mean profile shows no further trends. (Computational details and results are presented in [11].)

The surface rejuvenation frequency

The employment of the above-described calculation procedure requires that the numerical value of the mean residence time, τ, be known. Significantly a reasonable formulation for τ may be obtained on the basis of the surface rejuvenation model and experimental momentum transfer data. The adaptation of this model to momentum transfer is merely based on equations of the form

$$\frac{\partial u}{\partial \theta} = \nu\frac{\partial^2 u}{\partial y^2}$$
$$\begin{aligned} u &= 0 \quad \text{at } y = 0 \\ u &= U_i \quad \text{as } y \to \infty \\ u &= g(y) \text{ at } \theta = 0 \end{aligned} \tag{11}$$

where $g(y)$ is a truncated curve and the parameter U_i may be approximated by the bulk stream velocity, U_b. The coupling of the solution to this system of equations with the calculation procedure suggested by Harriott for random approach distances and uniform residence times leads to a relationship between the parameters $H^+\sqrt{(f/2)}$ and $H/\sqrt{(\nu\tau)}$ as shown in Fig. 1. (These calculations are identical to those for hH/K and $H/\sqrt{\alpha\tau}$ obtained on the basis of the similar heat transfer analysis.) Although calculations by Harriott [8] and Hanratty [7] have indicated that the selection of a time distribution is of secondary importance, the approach distance distribution has been shown to be quite significant [8, 16, 18]. With H^+ set equal to 5·0, the surface rejuvenation model gives rise to a functional relationship between τ and the Fanning friction factor, f. Hence, a reasonable value for τ

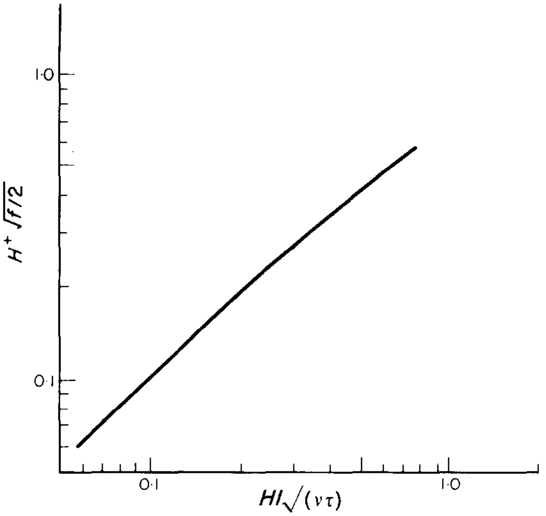

FIG. 1. Calculations for momentum transfer based on the surface rejuvenation model.

may be obtained for a given Reynolds number on the basis of the well known behaviour of f for turbulent flow of Newtonian fluids in smooth tubes.

DISCUSSION

Calculations for the mean dimensionless temperature profile obtained on the basis of the present analysis are compared with available experimental data for values of the Prandtl number between 0·02 and 14·3 in Fig. 2. The calculations are seen to be in excellent agreement with these experimental data. Calculations for T^+ based on the surface rejuvenation model are also compared with the relationship obtained on the basis of the elementary form of the surface renewal and penetration model, equation (2), in Fig. 2. These two surface renewal and penetration models lead to essentially identical relationships for the temperature profiles associated with values of the Prandtl number below 5·7. However, the curve resulting from equation (2) lies noticeably below the experimental data and calculations based on the surface rejuvenation model for $Pr = 14·3$. Hence, the effect on the transport flux of eddies not moving into direct contact with the surface, which is accounted for by the surface rejuvenation model, appears to become significant for values of the Prandtl number from 5·7 to 14·3 and greater. In this connection, calculations for the transport flux based on surface rejuvenation type models have been shown to diverge from calculations based on equation (3) for values of the Prandtl number of the order of 5·0 [16, 18].

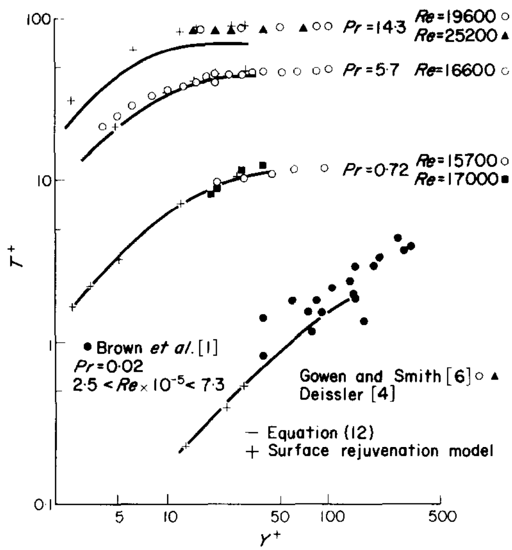

FIG. 2. Comparison of predicted temperature profiles with experimental data.

The importance of the unreplenished layer of fluid at the wall can be further demonstrated by a consideration of predicted temperature profiles associated with large values of the Prandtl number. Calculations for $(T - T_0)/(T_b - T_0)$ obtained on the basis of the surface rejuvenation model are compared with equation (1) for several values of the Prandtl number and Reynolds number in Fig. 3. Equation (1) lies well above the calculations based on the surface rejuvenation model, thus reflecting the inadequacy of the assumption that eddies move direct contact with the wall for large values of the Prandtl number.

Calculations for $(T - T_0)/(T_b - T_0)$ based on the semi-empirical model of Lin *et al.* [10] are also shown in Fig. 3. The model is based on an empirical formulation for the eddy diffusivity and has been reported to correlate experimental mass transfer data for large values of the Schmidt number. Although a basic consistency exists between the calculations based on the surface rejuvenation model and the model by Lin *et al.*, computations based on the former are seen to be slightly dependent on the Reynolds number whereas calculations based on the latter are essentially independent of the Reynolds number. The Reynolds number dependency indicated by the calculations obtained on the basis of the surface rejuvenation model can be shown to be reasonable by a consideration of $\partial T/\partial y^+$ at the wall. From the definition of the mean coefficient of heat transfer, an expression can be written for $\partial T/\partial y^+|_0$ as

$$\left.\frac{\partial\left(\dfrac{T - T_0}{T_b - T_0}\right)}{\partial y^+}\right|_0 = \frac{Nu}{Re\sqrt{(f/2)}}. \qquad (12)$$

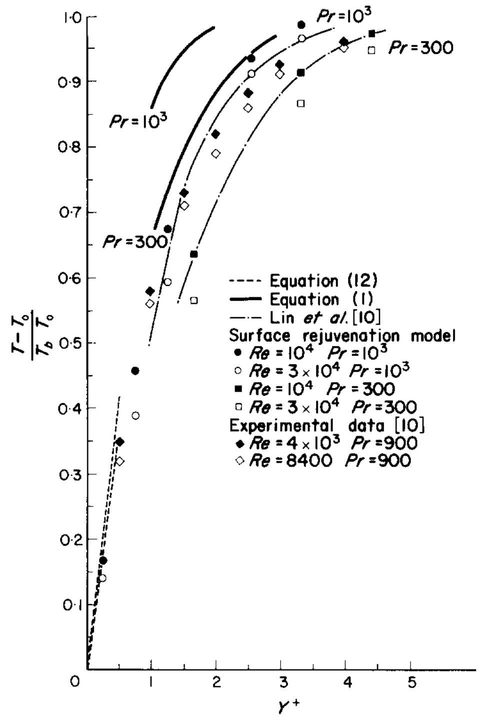

FIG. 3. Temperature profiles for high *Pr*.

Since the Nusselt number is essentially proportional to $f/2 \cdot Re$ for a given value of the Prandtl number [9], $\partial T/\partial y^+|_0$ is approximately proportional to $\sqrt{(f/2)}$. The Reynolds number dependency indicated by calculations for T based on the present model are shown in Fig. 3 to be consistent with equation (12). Further, interferometric measurements of concentration profiles for a Schmidt number of 900 obtained by Lin *et al.* are shown in Fig. 3 for values of the Reynolds number equal to 4000 and 8400. (Computations based on the present analysis for $Pr = 10^4$ with $Re = 10^4$ and $Re = 3 \times 10^4$ are equivalent to calculations for $Pr = 900$ with $Re = 7000$ and $2{\cdot}7 \times 10^4$, respectively.) Though some inconsistency in the data exists, the present analysis is in good agreement with these data for values of y^+ less than 2·0.

CONCLUDING REMARKS

The surface rejuvenation model has been found to successfully correlate experimental temperature profile data associated with turbulent tube flow for fluids with a wide range of Prandtl numbers. The formulation of the model is based on the hypotheses that the turbulent heat transfer is dependent upon (1) the mean approach distance as characterized by the rejuvenation process, (2) the molecular transport to fluid within the wall region, and (3) the mean frequency of rejuvenation. In this analysis the contribution of the unreplenished layer of fluid is considered and shown to be of primary concern for large values of the Prandtl number. For values of the

Prandtl number of the order of 5·0 and less the thermal resistance offered by this layer of fluid becomes negligible such that the basic form of the surface renewal and penetration model becomes adequate.

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PROFILS DE TEMPÉRATURE DANS DES FLUIDES EN ÉCOULEMENT TURBULENT ET À NOMBRE DE PRANDTL ÉLEVÉ

Résumé—Harriot a développé un modèle de transport turbulent qui coordonne les résultats expérimentaux de nombres de Nusselt moyens pour des fluides dans une large gamme de nombres de Prandtl. Ce modèle de renouvellement en surface est adapté ici à l'étude des profils de température ou de concentration pour un écoulement turbulent dans un tube en s'intéressant plus particulièrement aux fluides à nombre de Prandtl (Schmidt) élevé.

TEMPERATURPROFILE FÜR TURBULENTE STRÖMUNG VON FLÜSSIGKEITEN MIT HOHER PRANDTL-ZAHL.

Zusammenfassung—Harriot hat ein umfassendes Modell für den turbulenten Transport entwickelt, das es erlaubt, experimentelle Daten für die mittlere Nusselt-Zahl bei Flüssigkeiten in einem weiten Bereich von Prandtl-Zahlen zu erfassen. Dieses "Oberflächen-Verjüngungs-Modell" wird hier angepasst, um Temperatur- oder Konzentrationsprofile für turbulente Rohrströmung von Flüssigkeit unter besonderer Betonung hoher Prandtl-(Schmidt)-Zahlen zu untersuchen.

ТЕМПЕРАТУРНЫЕ ПРОФИЛИ ТУРБУЛЕНТНОГО ТЕЧЕНИЯ ЖИДКОСТЕЙ С БОЛЬШИМ ЧИСЛОМ ПРАНДТЛЯ

Аннотация—Харриотт разработал модель турбулентного переноса, которая может обобщать экспериментальные данные для среднего числа Нуссельта для жидкостей с большим диапазоном чисел Прандтля. В настоящей работе эта модель применяется для изучения температурных или концентрационных профилей при турбулентном течении в трубе, особенно жидкостей с большим числом Прандтля (Шмидта).